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The Genus of the Product of a Group with an Abelian Group

TOMAŽ PISANSKI

AND

THOMAS W. TUCKER

The genus of the direct product $G \times A$ of an arbitrary finite group G and a finite abelian group A is determined for many G and most abelian groups of sufficiently large rank. The computation uses the Jungerman and White current graphs for abelian groups, together with some observations about the rank of $G \times A$ that may be of independent interest.

1. INTRODUCTION

The *genus of a group G*, denoted $\gamma(G)$, is the minimum genus of any orientable surface containing an embedding of some Cayley graph for G. White [8] initiated the study of the genus of a group and together with Jungerman [4] computed the genus of 'most' abelian groups. For more information about the genus of a group, the reader is referred to Gross and Tucker [3].

This paper began with the study of hamiltonian groups, that is groups of the form $Q \times A_r \times Z_2^s$, where Q is the quaternions, A_r is an odd order abelian group of rank r, and Z_2^s is the product of s copies of the cyclic group of order 2. Pisanski and White [7] have computed the genus of such groups for all $s > 0$ if $r > 5$ and for all but finitely many s for $r \leq 5$. We considered the unsolved cases for $r \leq 3$ in [6]. The cases for $r > 5$ and $s = 0$ entail quite different methods. By a careful re-examination of Jungerman and White's current graphs for abelian groups, we were able to construct minimum genus embeddings for $r \equiv 2 \bmod 4$ when $s = 0$. We realized these constructions applied also to finite groups of the form $G \times A$, where A is abelian. To obtain the full generality, however, we needed more information about special kinds of generating sets for $G \times A$. The important lemma is that

$$\text{rank}(G \times A) = \max\{\text{rank}(G), \text{rank}(\bar{G} \times A)\},$$

where $\bar{G}$ is the abelianization of G, namely $G/[G, G]$.

The second section of this paper presents sufficient conditions under which Jungerman and White's constructions can be used. These are applied to certain special cases of the product $G \times A$, including the case of $r \equiv 2 \bmod 4$ for hamiltonian groups. The rank of $G \times A$ is analysed in Section 3 to obtain the genus of more general products.

2. JUNGERMAN–WHITE REVISITED

If X is a minimal generating set for the group G and X has no elements of order 3, then an embedding for the Cayley graph $C(G, X)$ will have minimum genus for that graph if all faces are quadrilaterals. If G is an abelian group, the commutators in pairs of generators provide an abundance of potential quadrilateral faces. The current graphs of Jungerman and White [4] show how to use those commutators to construct all-quadrilateral embeddings in nearly every case where the numbers allow such an embedding; that is, where $(\text{rank}(G) - 2) \cdot |G|$ is divisible by 4 (here $|G|$ denotes the order of G). These current graphs are quite complex and fall into four

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separate cases, but a careful inspection of them (see [3], for example) reveals that not all the generators need commute with each other for the current graphs to work. Hence the same current graphs can be applied to arbitrary groups, as long as these groups have generating sets with the right properties. The following theorem gives those properties in two of the four cases:

THEOREM 2.1. *Let $\{x_1, \ldots, x_n\} = X$ be a generating set for the finite group G such that x_i and x_{i+1} commute for $i = 1, \ldots, r-1$ and x_r and x_1 commute. Suppose, in addition, that one of the following holds:*
(a) *$r \equiv 0 \bmod 4$ and there is a homomorphism $\phi: G \to Z_2$ such that $\phi(x_i) = 1$ for all i;*
(b) *$r \equiv 2 \bmod 4$ and there is a homomorphism $\phi: G \to Z_m$ for some odd $m > 2$ such that $\phi(x_i) = 1$, all i.*
Then the Cayley graph $C(G, X)$ has an orientable all-quadrilateral embedding.

PROOF. Read carefully Cases 2 and 3 in [3, pp. 256–257] or in [4]. Check that at any vertex of the current graphs the only edges present are labeled x_i and x_{i+1}, for some $i = 1, \ldots, r-1$, or x_r and x_1. Alternatively, check that in the rotation system for the dual voltage graph, consecutive edges at any vertex are labelled x_i and x_{i+1} or x_1 and x_r. Finally, for Case 2 check that the given quotient voltage graph has two vertices and that every edge leads from one vertex to the other, which means that the quotient group is Z_2 and that the image of each generator is the generator of 1 of Z_2. Similarly, for Case 3, check that the quotient group is Z_m, where m where m is odd, and that the image of each generator x_i is generator 1 of Z_m. □

The remaining two cases of Jungerman and White involve more complicated current graphs and consequently more complicated commutator relations among the generators. A convenient way [5] of describing which pairs of generators commute is a *commutator graph:* Given a generating set $\{x_1, \ldots, x_r\} = X$ for a group G, define a graph with vertex set X and with an edge between x_i and x_j if and only if x_i and x_j commute. Thus the initial condition of Theorem 2.1 is that the commutator graph for the generating set X contains a hamiltonian cycle.

THEOREM 2.2. *Let $X = \{x_1, \ldots, x_r\}$ r odd and $r > 3$, be a generating set for the group G such that the commutator graph for X includes the cycle $x_1, x_2, \ldots, x_{r-1}, x_1$ and edges between x_r and x_j, $r-4 \leq j \leq r-1$. Suppose, in addition, that one of the following holds for some odd $m > 2$:*
(a) *there is a homomorphism $\phi: G \to Z_2 \times Z_2 \times Z_m$ such that $\phi(x_i) = (1, 1, 0)$ for i odd, $i \neq r$, that $\phi(x_i) = (0, 1, 0)$ for i even, and that $\phi(x_r) = (0, 0, 1)$;*
(b) *there is a homomorphism $\phi: G \to Z_4 \times Z_m$ such that $\phi(x_i) = (2, 0)$ for i odd, $i \neq r$, that $\phi(x_i) = (1, 0)$ for i even, and that $\phi(x_r) = (0, 1)$.*
They the Cayley graph $C(G, X)$ has an orientable, all-quadrilateral embedding.

PROOF. A careful inspection of the rotation system for the voltage graph of Case 4 in [3, pp. 259–261] reveals that the only consecutive edges are x_i and x_{i+1}, $i = 2, \ldots, r-1$ or x_2 and x_r or x_1 and x_j, $r-3 \leq j \leq r$. (Alternatively, check the corresponding current graph; note, however, that the labelling of the current graph is in error and should be revised—the labels $x_1, x_2, \ldots, x_r$ should be $x_r, x_{r-1}, \ldots, x_1$.) If the generators $x_1, x_2, \ldots, x_r$ are relabelled $x_2, \ldots, x_r, x_1$ (i.e. cyclically shift labels down one and take x_1 to x_r), the commutator graph is as described in the hypothesis of this theorem. Similarly, an inspection of the rotation system for the voltage graph in Case 5 of [3] reveals that the only consecutive edges are x_i and x_{i+1}, $i = 2, \ldots, r-5$, or x_{r-4} and x_{r-1}

or x_{r-1} and x_r or x_r and x_{r-3} or x_{r-3} and x_{r-2} or x_{r-2} and x_2 or x_1 and x_j, $r-3 \leq j \leq r$. Relabel generators as follows: interchange x_{r-1} and x_{r-3}, interchange x_{r-2} and x_{r-4}, then cyclically shift all labels down one and take x_1 to x_r. The resulting commutator graph is again as described in the hypothesis of this theorem.

Finally, the homomorphism ϕ is found, in each case, by seeing what the quotient group is and where the edges x_i lead in the quotient Cayley graph. Note that our relabelling interchanges odd and even and takes x_1 to x_r. □

Theorems 2.1 and 2.2 provide the upper bound

$$\gamma(G) \leq 1 + |G|\,(r-2)/4$$

for any group G of order $|G|$ having a generating set $x_1, \ldots, x_r$ satisfying the hypotheses of either Theorem 2.1 or 2.2. A lower bound is provided by the following:

THEOREM 2.3. *Let G be any finite group having a homomorphism onto an abelian group having canonical form*

$$Z_{m_1} \times Z_{m_2} \times \cdots \times Z_{m_r},$$

where $r > 1$, $m_1 > 3$, *and* $m_i \mid m_{i+1}$, $1 \leq i \leq r-1$. *Then* $\gamma(G) \geq 1 + |G|(r-2)/4$.

PROOF. Let X be a generating set for G and let s be the number of generators in X of order at least 4, let t_3 be the number of order 3, and let t_2 be the number of order 2. The proof follows from the computations in the proof of the analogous Theorem 6.1.2 in [3] or Theorem 1 in [4], as long as we can show that $s + t_2 \geq r$ and $s + t_3 \geq r$. Both proofs show that these two inequalities hold for any generating set for an abelian group having the given canonical form. Since such an abelian group is a quotient group of G, the same inequalities must also hold for any generating set for G ($s + t_2$ for the generating set X of G is greater than or equal to $s + t_2$ for the image-generating set in the abelian quotient group; similarly for $s + t_3$). □

Let A be an abelian group of rank r and order $|A|$. Call A a *JW abelian group* if the following hold:
(1) there are no Z_2 or Z_3 factors in the canonical form for A;
(2) $(r-2)\,|A| \equiv 0 \bmod 4$ and $r > 3$;
(3) $|A|$ is not a power of 2 if $r \neq 0 \bmod 4$.
The main result of Jungerman and White is that $\gamma(A) = 1 + |A|(r-2)/4$ if A is a JW abelian group. The main purpose of this paper is to show that

$$\gamma(G \times A) = 1 + |G \times A|\,(r-2)/4$$

for most groups G, where A is a JW abelian group of rank r at least twice the rank of G. The following two theorems are two examples of this type of result.

THEOREM 2.4 (*hamiltonian groups*). *Let Q be the quaternion group and let A be a JW abelian group of odd order and rank* $r > 2$. *Then*

$$\gamma(Q \times A) = 1 + |Q \times A|\,(r-2)/4.$$

PROOF. Since A has odd order, we must have $r \equiv 2 \bmod 4$ and $r > 5$. Let $a_1, \ldots, a_r$ be a 'canonical' generating set for A; that is, A is the direct product of the cyclic subgroups generated by $a_1, \ldots, a_r$ and the orders m_i for a_i, $i = 1, \ldots, r-1$, satisfy $m_i \mid m_{i+1}$. Denote the elements of Q, as usual, by ± 1, $\pm i$, $\pm j$ and $\pm k$. Then in $Q \times A$

let $x_1 = (i, a_1 + a_r)$, $x_2 = (1, a_2 + a_r)$, $x_3 = (j, a_3 + a_r)$, $x_4 = (1, a_4 + a_r)$, $x_5 = (1, a_5 + a_r), \ldots, x_{r-1} = (1, a_{r-1} + a_r)$, $x_r = (1, a_r)$. Then $x_1, \ldots, x_r$ generate $Q \times A$ and satisfy the commutator hypotheses and condition (b) of Theorem 2.1, where $\gamma: Q \times A \to Z_{m_r}$ is a projection onto the canonical factor of A generated by a_r. □

THEOREM 2.5. *Let G be a group whose commutator subgroup $[G, G]$ is all of G. Let A be a JW abelian group of rank r at least twice the rank of G. Then $\gamma(G \times A) = 1 + |G \times A|\,(r-2)/4$.*

PROOF. As in the proof of Theorem 2.4, let $a_1, \ldots, a_r$ be a canonical generating set for A.

Suppose r is even. Let $b_i = a_i + a_r$ for $i < r$ and $b_r = a_r$. Observe that if $r \equiv 2 \bmod 4$, then the order m_r of a_r has an odd factor m and hence there is a homomorphism of A onto Z_m taking b_i to 1 for all i. Also observe that if $r \equiv 0 \bmod 4$, then $|A|$ must be even; therefore m_r is even and there is a homomorphism of A onto Z_2 taking b_i to 1 for all i.

Suppose instead that r is odd. Then $|A|$ is divisible by 4 by property (2) of JW abelian groups. Since $|A|$ also must have an odd factor by (3), it follows that at least one of the two following cases holds:
(a) one of the three numbers m_{r-2}, m_{r-1}, m_r has an odd factor and the other two are even;
(b) one of the two numbers m_{r-1}, m_r has an odd factor and the other is divisible by 4.
Without loss of generality, we can assume that the number with the odd factor is m_r (if not, simple re-order the last factors of A). In case (a) let $b_i = a_i + a_{r-2} + a_{r-1}$ for i odd, $i < r-2$; let $b_i = a_i + a_{r-1}$ for i even, $i < r-1$; let $b_{r-2} = a_{r-2} + a_{r-1}$; let $b_{r-1} = a_{r-1}$; and let $b_r = a_r$. In case (b) let $b_i = a_i + 2a_{r-1}$ for i odd, $i < r$; let $b_i = a_i + a_{r-1}$ for i even, $i < r-1$; let $b_{r-1} = a_{r-1}$; and let $b_r = a_r$. It is easily verified that $b_1, \ldots, b_r$ generate A and that the homomorphism ϕ required in cases (a) and (b) of Theorem 2.2 exists.

Let $g_1, \ldots, g_s$ be a generating set for G, where s is the rank of G. Define x_i in $G \times A$ for $i = 1, \ldots, r$ to be (e, b_i) for i even and $(g_{(i+1)/2}, b_i)$ for i odd (e denotes the identity element of G). Then $x_1, \ldots, x_r$ satisfy the commutator conditions in Theorems 2.1 and 2.2, and the required homomorphism ϕ is obtained by first projecting onto A. It remains to show that $x_1, \ldots, x_r$ generate $G \times A$. The commutator subgroup $[G, G]$ of G is generated by the commutators in $g_1, \ldots, g_s$ and their conjugates. Since A is abelian, all commutators in $x_1, \ldots, x_r$ and their conjugates have trivial A co-ordinate. Thus $x_1, \ldots, x_r$ generate $[G, G] \times \{0\}$. Since $[G, G] = G$ and the A co-ordinates of $x_1, \ldots, x_r$ generate A, it follows that $x_1, \ldots, x_r$ generate $G \times A$.

We conclude that $x_1, \ldots, x_r$ is a generating set for $G \times A$ satisfying the hypotheses of Theorem 2.1 or 2.2. Thus $\gamma(G \times A) \leq 1 + |G \times A|(r-2)/4$. By Theorem 2.3, $\gamma(G \times A) \geq 1 + |G \times A|\,(r-2)/4$. □

COROLLARY. *Let G be a finite simple group, and let A be a JW abelian group of rank $r \geq 4$. Then $\gamma(G \times A) = 1 + |G \times A|\,(r-2)/4$.*

PROOF. Since G is simple, $G = [G, G]$. By the classification of finite simple groups [2], the rank of G is 2. The corollary thus follows directly from Theorem 2.5. □

Theorem 2.4 is the result about hamiltonian groups mentioned in the introduction of this paper and referred to in [6]. It would be nice to handle such hamiltonian groups for values of r other than $r \equiv 2 \bmod 4$, especially since the abelianization of $Q \times A$ has order divisible by 4. One might hope that the Z_2 factors in the abelianization of Q

would help out in constructing the homomorphism ϕ when $r \neq 2 \bmod 4$. Unfortunately, any generating set having the required ϕ cannot satisfy the commutator conditions in Theorems 2.1 and 2.2, since every generator would need to have a Q co-ordinate other than ± 1.

3. More General Products

Theorem 2.5 computes the genus of $G \times A$ for most abelian groups A of sufficiently large rank and all groups G with trivial abelianization. Theorem 2.4 indicates that the restriction on the abelianization can be loosened if $|G/[G, G]|$ is relatively prime to $|A|$ (in Theorem 2.4, $Q/[Q, Q] = Z_2 \times Z_2$ and $|A|$ is odd). In this section, we loosen that restriction further by considering carefully generating sets for $G \times A$.

First we observe that finding a generating set for $G \times A$ reduces to finding one for $G/[G, G] \times A$. This was implicit in Theorem 2.5, but to be complete we prove the following lemma. For convenience, denote by $\bar{G}$ the abelianization $G/[G, G]$ and denote by $\bar{g}$ the image in $G/[G, G]$ of any element $g \in G$.

Lemma 3.1. *The elements $(g_1, a_1), \ldots, (g_n, a_n)$ generate $G \times A$ if and only if $g_1, \ldots, g_n$ generate G and $(\bar{g}_1, a_1), \ldots, (\bar{g}_n, a_n)$ generate $\bar{G} \times A$.*

Proof. Suppose that $g_1, \ldots, g_n$ generate G and $(\bar{g}_1, a_1), \ldots, (\bar{g}_n, a_n)$ generate $\bar{G} \times A$. Since $\bar{G} \times A$ is the quotient of $G \times A$ by $[G, G] \times \{0\}$, it suffices to show that $[G, G] \times \{0\}$ is contained in the subgroup generated by $(g_1, a_1), \ldots, (g_n, a_n)$. But $[G, G]$ is generated by commutators in $g_1, \ldots, g_n$ and their conjugates, and any commutator in $(g_1, a_1), \ldots, (g_n, a_n)$ or conjugate thereof has trivial A co-ordinate since A is abelian. Thus $[G, G] \times \{0\}$, and hence $G \times A$, is generated by $(g_1, a_1), \ldots, (g_n, a_n)$. The converse follows by projecting the generating set $(g_1, a_1), \ldots, (g_n, a_n)$ onto G and onto $\bar{G} \times A$. □

Next, we observe that finding a generating set for an abelian group B reduces to finding one for each of the Sylow subgroups of B, because B is the direct product of its Sylow subgroups. To make this explicit, let B_p denote the p-Sylow subgroup of B corresponding to a prime p dividing $|B|$. For $x \in B$, let x_p denote the projection of x onto the factor B_p in the direct product factorization of B into Sylow subgroups. If $X \subset B$, let X_p be the set of x_p, $x \in X$.

Lemma 2.2. *The set X generates the abelian group B if and only if X_p generates B_p for each prime p dividing $|B|$.*

Proof. Suppose that X_p generates B_p for each p. Let p^α be the largest power of p dividing $|B|$. Let $e = |B|/p^\alpha$. For any x, $(x^e)_q = 0$ for all $q \neq p$ so $x^e = (x^e)_p = (x_p)^e$. Since e is relatively prime to p, it follows that x_p is in the cyclic group generated by x. Thus for each p, B_p is contained in the subgroup of B generated by X and hence X generates B. The converse follows by projecting the generating set X onto the factor B_p. □

Finally, we observe that finding a generating set for an abelian p-group (that is, a group of order a power of p) reduces to finding one for an elementary abelian p-group (that is, a product of cyclic groups Z_p). This follows from elementary facts about the Frattini subgroup of a p-group (see Gorenstein [1], for example) but the proof is easy and self-contained so we give it anyway. If B is an abelian group, let pB denote the subgroup of elements of the form pb, $b \in B$.

LEMMA 2.3. *Let B be an abelian p-group. Then X generates B if and only if the image of X generates the quotient group B/pB.*

PROOF. Clearly, if X generates B, its image generates B/pB. Conversely, suppose X does not generate B. Let H be the largest proper subgroup of B containing X. We claim that pB is contained in H, and hence that the image of H is a proper subgroup of B/pB, which means the image of X does not generate B/pB. Suppose $pb \notin H$ for some $b \in B$. By the maximality of H, it follows that pb together with H generate B. Thus $b = h + apb$ for some $h \in H$ and some integer a. Therefore $(1 - ap)b$ is in H. Since $1 - ap$ is relatively prime to p, it follows that b is also in H, contradicting the assumption that $pb \notin H$. □

Notice that every element of B/pB had order p and hence B/pB can be viewed as a vector space over Z_p. Thus the following theorem really boils down to a statement about spanning sets of a vector space.

THEOREM 3.4. *Suppose that $g_1, \ldots, g_n$ generate G and that $\operatorname{rank}(\bar{G} \times A) \leq n$, where A is an abelian group. Then there exist $a_1, \ldots, a_n$ in A such that $(g_1, a_1), \ldots, (g_n, a_n)$ generate $G \times A$.*

PROOF. By Lemmas 3.1–3.3, we can assume that G and A are elementary abelian p-groups and hence vector spaces over Z_p. Let $k = \operatorname{rank} G$ and $m = \operatorname{rank}(G \times A)$; thus G and $G \times A$ as vector spaces over Z_p have dimension k and m respectively. Since the set $\{g_1, \ldots, g_n\}$ spans G, it contains a basis which, without loss of generality, we assume is $\{g_1, \ldots, g_k\}$. Let $b_1, \ldots, b_{m-k}$ be any basis for A. Now define $a_i = 0$ for $1 \leq i \leq k$, $a_i = b_{i-k}$, $k < i \leq m$. If $n > m$, define $a_i = 0$ for $m < i \leq n$. Then $(g_1, a_1), \ldots, (g_n, a_n)$ form the desired generating set for $G \times A$. □

COROLLARY. *For any finite group G and finite abelian group A,* $\operatorname{rank}(G \times A) = \max\{\operatorname{rank}(G), \operatorname{rank}(\bar{G} \times A)\}$.

One might guess that for arbitrary groups G and H $\operatorname{rank}(G \times H) = \max\{\operatorname{rank}(G), \operatorname{rank}(H), \operatorname{rank}(\bar{G} \times \bar{H})\}$. This works more often than one might expect, but it is false in general. If G is a group of rank 2 with trivial $\bar{G}$, this equality would say that $\operatorname{rank}(G^s) = 2$, where G^s is the product of s copies of G, $s > 0$. Suppose $s > |G|^2$ and x and y are any two elements of G^s. By the pigeonhole principle, there exist i and j, $i \neq j$, such that $x_i = x_j$ and $y_i = y_j$, where x_i denotes the co-ordinate of x in the ith factor of the product G^s. It follows that x and y cannot generate G^s and so $\operatorname{rank}(G^s) \neq 2$.

The following is the promised genus theorem for more general products with an abelian group A. Basically, it computes the genus of $G \times A$ when $\bar{G} \times A$ is JW except that conditions (2) and (3) in the definition of a JW abelian group hold for $|A|$ rather than $|\bar{G} \times A|$.

THEOREM 3.5. *Suppose that G and A are finite groups, that A is abelian, and that $G \times A$ has rank r at least twice the rank of G. Suppose further that:*
(1) there are no Z_2 or Z_3 factors in the canonical form of $\bar{G} \times A$;
(2) $(r-2)\,|A| \equiv 0 \bmod 4$;
(3) $|A|$ is not a power of 2 if $r \not\equiv 0 \bmod 4$.
Then $\gamma(G \times A) = 1 + |G \times A|\,(r-2)/4$.

PROOF. Let $n = \operatorname{rank}(G)$ and $s = \operatorname{rank}(A)$. Let $a_1, \ldots, a_s$ be a canonical generating set for A (that is, A is the direct product of the cyclic subgroups generated by $a_1, \ldots, a_s$ and the orders $m_1, \ldots, m_s$ of $a_1, \ldots, a_s$ satisfy $m_1 \mid m_2 \mid \cdots \mid m_s$). We claim there is a generating set $y_1, \ldots, y_r$ for $G \times A$ such that $y_{r-i} = (e, a_{s-i})$ for $0 \leq i < r - n$, where e is the identity of G. Then the generating set $x_1, \ldots, x_r$ required by Theorem 2.4 or 2.5 is constructed as follows. First re-order $y_1, \ldots, y_{2n}$ so that $y_1, \ldots, y_n$ alternate with $y_{n+1}, \ldots, y_{2n}$. Since y_i has trivial G co-ordinate for $i > n$, this means that the commutator conditions of Theorem 2.4 or 2.5 are satisfied. Then add the necessary combinations of a_{s-2}, a_{s-1}, a_s to the A co-ordinates of the other generators, as in the proofs of Theorem 2.4 and 2.5, so that the required homomorphism ϕ exists. Conditions (2) and (3) in the hypotheses of this theorem guarantee that m_{s-2}, m_{s-1}, m_s have the needed parity or odd factor; the property that $y_{r-i} = (e, a_{s-i})$ for $i < r - n$ guarantees that a_{s-2}, a_{s-1}, a_s are free to add to the A co-ordinates of the other generators (notice that $r - n \geq 3$ unless $n = 2$ and $r = 4$, and when $r = 4$ there is no problem because only a_{s-1} and a_s are needed in the case $r \equiv 0 \bmod 4$).

Thus if the claimed generating set exists, Theorem 2.4 or 2.5 can be applied and we conclude that $\gamma(G \times A) \leq 1 + |G \times A|\,(r-2)/4$. By Theorem 3.4, $\operatorname{rank}(\bar{G} \times A) = r$. Since $\bar{G} \times A$ has no Z_2 or Z_3 factors in its canonical form, we can apply Theorem 2.3 and conclude that $\gamma(G \times A) \geq 1 + |G \times A|\,(r-2)/4$.

It remains to show that the claimed generating set $y_1, \ldots, y_r$ exists. Let A_i, $1 \leq i \leq s$ be the subgroup of A generated by $a_1, \ldots, a_i$, and let $A_0 = \{0\}$. Define r_i to be the rank of $\bar{G} \times A_i$. Then $r_0 = \operatorname{rank} \bar{G} \leq n$, $r_s = \operatorname{rank}(\bar{G} \times A) = \operatorname{rank}(G \times A) = r$, and $r_i \leq r_{i+1} \leq r_i + 1$. Thus there is a last integer k such that $r_k = n$; in fact, $0 \leq k \leq s - (r - n) = (s + n) - r$. For some prime p, the rank of $\bar{G} \times A_k$ equals the rank of the p-Sylow subgroup of $\bar{G} \times A_k$. Since $r_{k+1} > r_k$ and $\bar{G} \times A_{k+1} = \bar{G} \times A_k \times Z_{m_{k+1}}$, it follows that p divides m_{k+1}. Therefore p divides m_i for all $i > k$, and hence $r_{i+1} = r_i + 1$ for all $i \geq k$. In particular, $r = n + (s - k)$. By Theorem 3.4, $G \times A_k$ has a generating set of n elements $y_1, \ldots, y_n$ because $\operatorname{rank}(\bar{G} \times A_k) = \operatorname{rank}(G) = n$. Since $r = n + (s - k)$, a generating set for $G \times A$ can then be obtained by appending to $y_1, \ldots, y_n$ the $s - k$ elements $(e, a_{k+1}), \ldots, (e, a_s)$. The resulting generating set has the desired properties. □

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References

1. D. Gorenstein, *Finite Groups*, Chelsea, New York, 1980.
2. D. Gorenstein, *Finite Simple Groups*, Plenum Press, New York, 1982.
3. J. L. Gross and T. W. Tucker, *topological Graph Theory*, Wiley-Interscience, New York, 1987.
4. M. Jungerman and A. T. White, On the genus of finite abelian groups. *Europ. J. Comb.*, **1**, (1980), 243–251.
5. T. Pisanski, Universal commutator graphs, *Discr. Math.*, to appear.
6. T. Pisanski and T. W. Tucker, The genus of low rank hamiltonian groups, *Discr. Math.*, to appear.
7. T. Pisanski and A. T. White, Nonorientable embeddings of groups, *Europ. J. Comb.*, **9** (1988), 445–461.
8. A. T. White, *Graphs, Groups, and surfaces*, revised edn, North Holland, Amsterdam, 1984.

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Tomaž Pisanski
Univerza v Ljubljani, 61111 Ljubljana, Yugoslavia
and
Thomas W. Tucker

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